L23 Conditional tests with one-sided H_a

1. Conditional tests with one-sided H_a.

   (1) α-level UMP with one-sided H_a.
   Recall: if $\theta_1 < \theta_2 \Longrightarrow \frac{f(x;\theta_2)}{f(x;\theta_1)}$ is an increasing function of $T(X)$, then

   For $H_0: \theta \le \theta_0$ versus $H_a: \theta > \theta_0$
   $$\phi(T) = \begin{cases} 1 & T > c \\ r & T = c \\ 0 & T < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(T)] = \alpha$$
   is α-level UMP

   For $H_0: \theta \ge \theta_0$ versus $H_a: \theta < \theta_0$
   $$\phi(T) = \begin{cases} 1 & T < c \\ r & T = c \\ 0 & T > c \end{cases} \quad \text{with } E_{\theta_0}[\phi(T)] = \alpha$$
   is α-level UMP

   (2) Nuisance parameter
   $f(x;\, \theta,\, \tau) = g_1(\theta,\, \tau) g_2(x)\, e^{r(\theta)t(x)}\, e^{\tau s(x)}$ where $r(\theta)$ is an increasing function of θ.
   Then $T(X)$ is sufficient and complete for θ and $S(X)$ is sufficient and complete for τ.
   By 1-1 mapping $y = y(x) = \begin{pmatrix} t \\ s \\ y_* \end{pmatrix} \Longleftrightarrow x = x(y)$ with $J(t,\, s,\, ,\, y_*) = \text{abs}\left|\frac{\partial\, x}{\partial\, y}\right|$,
   $f_Y(t,\, s,\, y_*;\, \theta,\, \tau) = g_1(\theta,\, \tau) g_2(t,\, s,\, y_*)\, e^{r(\theta)t}\, e^{\tau s} J(t,\, s,\, y_*)$. So
   $f_{(T,\,S)}(t,\, s;\, \theta,\, \tau) = g_1(\theta,\, \tau) e^{r(\theta)t} e^{\tau s} h(t,\, s)$, $f_T(t;\, \theta,\, \tau) = g_1(\theta,\, \tau) e^{r(\theta)t} h_t(t, \tau)$,
   $f_S(s;\, \theta,\, \tau) = g_1(\theta,\, \tau) e^{\tau s} h_s(s,\, \theta)$ and $f_{T|S}(t;\, \theta,\, s) = e^{r(\theta)t} \frac{h(t,\, s)}{h_s(s,\, \theta)}$.
   Thus when $\theta_1 < \theta_2$,
   $$\frac{f(X;\, \theta_2,\, \tau)}{f(X;\, \theta_1,\, \tau)} = \frac{g_1(\theta_2,\, \tau)}{g_1(\theta_1,\, \tau)}\, e^{[r(\theta_2) - r(\theta_1)]T(X)} = \frac{f_T(T;\, \theta_2,\, \tau)}{f_T(T;\, \theta_1,\, \tau)}$$
   is an increasing function of $T(X)$ for all τ; and
   $$\frac{f_{T|S}(t;\, \theta_2,\, s)}{f_{T|S}(t;\, \theta_1,\, s)} = e^{[r(\theta_2) - r(\theta_1)]T(X)}\, \frac{h_s(s\, \theta_1)}{h_s(s,\, \theta_2)}$$
   is an increasing function of $T(X)$ for all S.

   (3) Conditional tests one-sided H_a.
   With $f(x;\, \theta,\, \tau)$ in (2)

   For $H_0: \theta \le \theta_0$ versus $H_a: \theta > \theta_0$
   $$\phi(T) = \begin{cases} 1 & T > c \\ r & T = c \\ 0 & T < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(T)|S] = \alpha$$
   is conditional α-level UMP

   For $H_0: \theta \ge \theta_0$ versus $H_a: \theta < \theta_0$
   $$\phi(T) = \begin{cases} 1 & T < c \\ r & T = c \\ 0 & T > c \end{cases} \quad \text{with } E_{\theta_0}[\phi(T)|S] = \alpha$$
   is conditional α-level UMP

2. Statistics other than T

(1) $U = U(T, S)$

If $U = U(T, S)$ is an increasing function of T for all S, then $\frac{f_{T|S}(t;\theta_2,s)}{f_{T|S}(t;\theta,s)}$ is an increasing function of T for all S. Thus

For $H_0 : \theta \le \theta_0$ versus $H_a : \theta > \theta_0$

$$\phi(U) = \begin{cases} 1 & U > c \\ r & U = c \\ 0 & U < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(U)|S] = \alpha$$

is conditional α-level UMP

For $H_0 : \theta \ge \theta_0$ versus $H_a : \theta < \theta_0$

$$\phi(U) = \begin{cases} 1 & U < c \\ r & U = c \\ 0 & U > c \end{cases} \quad \text{with } E_{\theta_0}[\phi(U)|S] = \alpha$$

is conditional α-level UMP

(2) Selecting $U = U(T, S)$

Suppose the distribution of U selected in (1) is free of τ when $\theta = \theta_0$, i.e., U is ancillary for τ when $\theta = \theta_0$. But S is sufficient and complete for τ. By Basu theorem, U and S are independent when $\theta = \theta_0$. Hence the conditional distribution of U given S is the distribution of U when $\theta = \theta_0$. Thus the condition on the tests in (1) becomes $E_{\theta_0}[\phi(U)] = \alpha$. Clearly there is no conditional distribution involved and hence the tests are α-level UMP tests.

Ex1: For $N(\mu, \sigma^2)$ with $\theta = \frac{\mu}{\sigma^2}$ and $\tau = \sigma^2$,

$$H_0 : \mu \le 0 \text{ vs } H_a : \mu > 0 \Longleftrightarrow H_0 : \theta \le 0 \text{ vs } H_a : \theta > 0.$$

Recall: $T = \sum X_i$ is sufficient and complete for θ and $S = \sum X_i^2$ is sufficient and complete for τ.

Let $U = \frac{T/n}{\sqrt{(S-T^2/n)/n(n-1)}}$. Then U is an increasing function of T for all S, and $U \overset{\theta=0}{\sim} t(n-1)$. Hence in

$$\phi(U) = \begin{cases} 1 & U > c \\ 0 & U \le c \end{cases}$$

$\alpha = E_{\theta=0}[\phi(U)] = P_{\theta=0}(U > c) = P(t(n-1) > c) \Longrightarrow c = t_\alpha(n-1)$. Therefore

$H_0 : \mu \le 0$ vs $H_a : \mu > 0$

Test statistic: $U = \frac{\overline{X}}{\sqrt{s^2/n}}$

Reject H_0 if $U > t_\alpha(n-1)$

is α-level conditional UMP test

L24 Conditional test with two-sided H_a

1. Conditional test with two-sided H_a

   (1) α-level UMP with two-sided H_a
   Recall: Under $f(x;\,\theta) = \exp\left[p(\theta) + q(x) + r(\theta)T(x)\right]$ for

$$H_0:\,\theta = \theta_0 \text{ versus } H_a:\,\theta \neq \theta_0$$

$$\phi(T) = \begin{cases} 1 & T < c_1 \text{ or } T > c_2 \\ r_i & T = c_i,\, i = 1,\, 2 \\ 0 & c_1 < T < c_2 \end{cases} \quad \text{with } E_{\theta_0}[\phi(T)] = \alpha \text{ and } E_{\theta_0}[T\phi(T)] = \alpha E_{\theta_0}(T)$$

   is α-level UMP.

   (2) Nuisance parameter
   Now assume $f(x;\,\theta,\,\tau) = \exp\left[p(\theta,\,\tau) + q(x) + r(\theta)T(x) + h(\tau)S(x)\right]$.
   Rewrite $f(x;\,\theta,\,\tau) = \exp\left[p(\theta,\,\tau) + q_*(\tau,\,x) + r(\theta)T(x)\right]$ and regard θ as parameter of interest only. By (1) for the H_0 and H_a there is

$$\phi(T) = \begin{cases} 1 & T < c_1 \text{ or } T > c_2 \\ r_i & T = c_i,\, i = 1,\, 2 \\ 0 & c_1 < T < c_2 \end{cases} \quad \text{with } E_{\theta_0}[\phi(T)] = \alpha \text{ and } E_{\theta_0}[T\phi(T)] = \alpha E_{\theta_0}(T)$$

   However $E_{\theta_0}[\phi(T)]$, $E_{\theta_0}[T\phi(T)]$ and $E_{\theta_0}(T)$ depend on nuisance parameter τ that needs to be "removed".

   (3) Conditional test
   Replace the distribution of T under $\theta = \theta_0$ by the conditional distribution of T given S under $\theta = \theta_0$. Because S is sufficient for τ, the nuisance parameter is successfully "removed". We therefore obtain conditional α-level UMP.

$$\phi(T) = \begin{cases} 1 & T < c_1 \text{ or } T > c_2 \\ r_i & T = c_i,\, i = 1,\, 2 \\ 0 & c_1 < T < c_2 \end{cases} \quad \text{with } E_{\theta_0}[\phi(T)|S] = \alpha \text{ and } E_{\theta_0}[T\phi(T)|S] = \alpha E_{\theta_0}(T|S)$$

2. Simplification

   (1) Using $U = U(T,\,S)$
   If $U = U(T,\,S)$ is an increasing function of T for all S, then there exist $c_1(S)$ and $c_2(S)$ such that

$$\begin{cases} T < c_1 \text{ or } T > c_2 \\ T = c_i,\, i = 1,\, 2 \\ c_1 < T < c_2 \end{cases} \iff \begin{cases} U < c_1(S) \text{ or } U > c_2(S) \\ U = c_i(S),\, i = 1,\, 2 \\ c_1(S) < U < c_2(S) \end{cases}$$

   So the conditional test can be written as

$$\phi(U) = \begin{cases} 1 & U < c_1(S) \text{ or } U > c_2(S) \\ r_i & U = c_i(S),\, i = 1,\, 2 \\ 0 & c_1(S) < U < c_2(S) \end{cases}$$

   with $E_{\theta_0}[\phi(U)|S] = \alpha$ and $E_{\theta_0}[U\phi(U)|S] = \alpha E_{\theta_0}(U|S)$

Comments: Clearly we now have many options when selecting U.

In $\phi(U)$, $c_i(S)$ can still be written as c_i since when determining c_i by $E_{\theta_0}[\phi(U)|S] = \alpha$ and $E_{\theta_0}[U\phi(U)|S] = \alpha E_{\theta_0}(U|S)$, c_i will be $c_i(S)$.

(2) A special U

If the distribution of U under $\theta = \theta_0$ is free of τ, then U is ancillary for τ under $\theta = \theta_0$. But S is sufficient and complete for τ. By Basu Theorem, the distributions of U and S under $\theta = \theta_0$ are independent. Hence the conditional distribution of U given S is the distribution of U when $\theta = \theta_0$. Thus the test becomes

$$\phi(U) = \begin{cases} 1 & U < c_1 \text{ or } U > c_2 \\ r_i & U = c_i,\ i = 1, 2 \\ 0 & c_1 < U < c_2 \end{cases} \quad \text{with } E_{\theta_0}[\phi(U)] = \alpha \text{ and } E_{\theta_0}[U\phi(U)] = \alpha E_{\theta_0}(U).$$

There is no conditional distribution involved and hence $\phi(U)$ is an α-level UMP.

3. An example

(1) The problem

Test $H_0 : \mu = \mu_0$ versus $H_a : \mu \neq \mu_0$ where μ is in $N(\mu, \sigma^2)$.

(2) Analysis

Let $\theta = \frac{\mu - \mu_0}{\sigma^2}$ and $\tau = \sigma^2$. Then the hypotheses become

$$H_0 : \theta = 0 \text{ versus } H_a : \theta \neq 0$$

Let $T = \sum(X_i - \mu_0)$ and $S = \sum(X_i - \mu_0)^2$. Then

$$f(x;\, \theta,\, \tau) = \exp\left[-\frac{n}{2}\ln(2\pi\tau) - \frac{n}{2}\theta^2\tau - \theta T - \frac{1}{2\tau}S\right].$$

Let $U = \frac{T/n}{\sqrt{(S - T^2/n)/n(n-1)}}$. Then U is an increasing function of T for all S.

But $U = \frac{\overline{X} - \mu_0}{\sqrt{s_x^2/n}} \overset{\theta=0}{\sim} t(n-1)$. So $\phi(U) = \begin{cases} 1 & U < c_1 \text{ or } U > c_2 \\ 0 & c_1 < U < c_2 \end{cases}$.

Note that $E_{\theta=0}[U\phi(U)] = \alpha E_{\theta=0}(U) = 0 \Longrightarrow c_1 = -c_2 \overset{def}{=\!=} c$.

So $\alpha = E_{\theta=0}[U\phi(U)] \Longrightarrow c = t_{\alpha/2}(n-1)$.

(3) Conclusion

$H_0 : \mu = \mu_0$ versus $H_a : \mu \neq \mu_0$
Test statistic $U = \frac{\overline{X} - \mu_0}{\sqrt{s_x^2/n}}$
Reject H_0 if $U < -t_{\alpha/2}(n-1)$ or $U > t_{\alpha/2}(n-1)$

is α-level UMP.